Lecture 10

Plan: 1) D~~iscuss pset~~/
~~misc. remarks~~;
BFS → Caratheodory's Theorem.

→ 2) Total unimodularity

#3) b) show if G factor critical, edmonds terminates at single vertex.

Strategy: show ODD at end of Edmonds is empty b/c G_{final} is factor critical. using (a).

Easier: ODD is TB minimizer for $G_{original}$! From class.

$\Rightarrow$ ODD $= \emptyset$ from part (a).

$\Rightarrow$ implies G_{final} is singleton, because $G_{original}$ is connected.

Quiz see Canvas announcement.

1) format: 2 hrs, 22 hr window for starting it, window starts 11:00 am Thurs Apr 1.

2) open note, no collaborators, no internet a art from Canvas.

3) is practice quiz already,
is assignment in canvas.

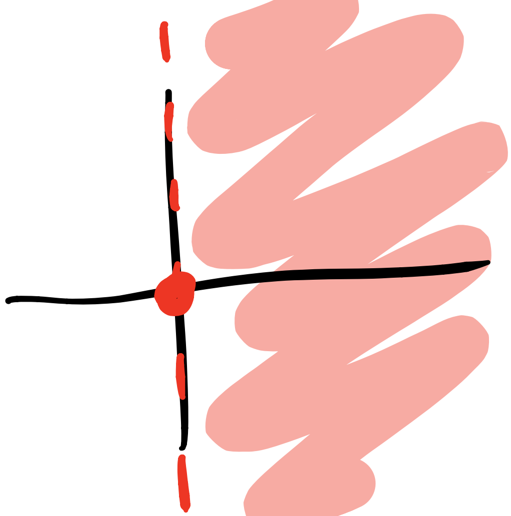
\# questions 3-4.

No lecture Apr 1.

#5) $\text{cone}(P) = \{0\} \cup \{x : \exists \lambda \text{ s.t. } \lambda x \in P\}$.

$= \{0\} \cup \{x : \exists \lambda > 0 \text{ s.t. } A\lambda x \leq b\}$

$= \{0\} \cup \{x : \exists \lambda > 0 \text{ s.t. } Ax \leq \lambda b\}$.



cone could be

$\{(x,y) : x > 0\}$

$\cup \{0\}$.

$P = \{x : \exists \lambda > 0 \text{ s.t. } Ax \leq \lambda b\}$

as projection of $\tilde{P}$

$$\tilde{P} = \{(x, \lambda) : \underbrace{Ax - \lambda b \le 0}, \lambda \geq 0\}$$

$$P = \tilde{P}_{n+1}$$

$$\tilde{A} = \begin{array}{c} \begin{array}{cc} \quad x \quad & \lambda \end{array} \\ \left[\begin{array}{c|c} A & -b \\ \hline 0 & -1 \end{array}\right] \end{array}$$

$$\tilde{P} = \{(x, \lambda) : \tilde{A}\,(x, \lambda) \le 0$$

$$< 0$$

$$\tilde{A} = \left[\begin{array}{c|c} A & -b \\ \hline 0 & -1 \end{array}\right] \begin{array}{l} \} I_< \longrightarrow b_i > 0 \\ \} I_= \longrightarrow b_i = 0. \\ \} I_> \longrightarrow b_i < 0 \\ \leftarrow I_< \end{array}$$

e.g. can combine I >.
and final inequality $-\lambda < 0$.

$$a_i^T x - b_i \lambda \leq 0 \quad b_i < 0$$

$$+ \ (-b_i)[-\lambda < 0]$$

$$\rightarrow \boxed{a_i^T x < 0 \quad \forall i : b_i < 0}$$

Misc remarks

aff(X)
conv(X)
cone(X)

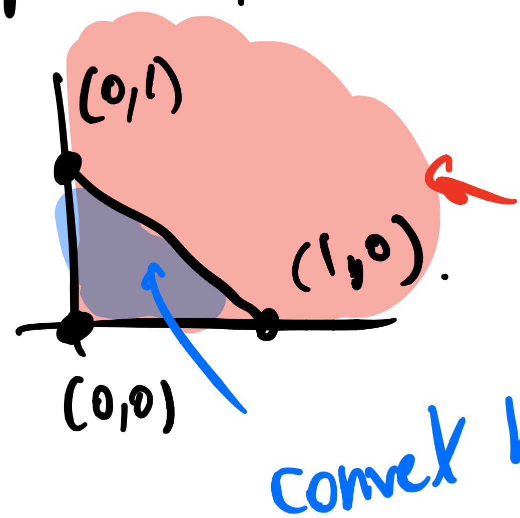
$$\text{aff}(X) = \sum x_i \lambda_i \quad : \sum \lambda_i = 1$$

$$\text{cone}(X) = \sum x_i \lambda_i \quad : \lambda_i \geqslant 0$$

$$\text{conv}(X) = \sum x_i \lambda_i \quad : \lambda_i \geqslant 0 \,,\ \sum \lambda_i = 1.$$

is $\mathrm{conv}(X) = \mathrm{cone}(X) \cap \mathrm{aff}(X)$?

NOT TRUE



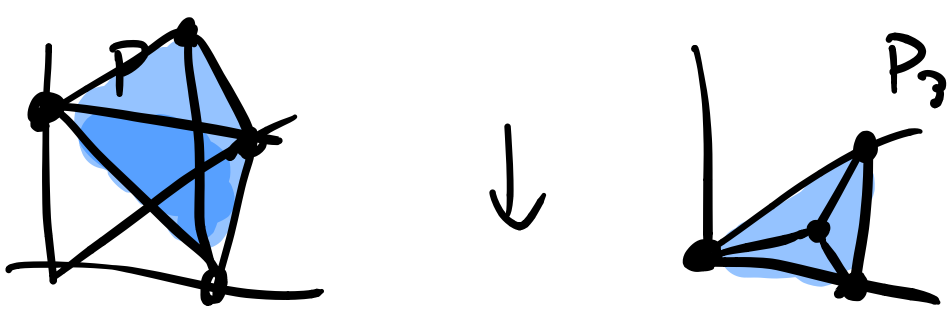
affine hull all of $\mathbb{R}^2$

affine $\cap$ conic = conic!

why not contradiction??

Misc.remark #2

$P = \{x : Ax \le b\}$ consider projection P_n



Know how to write $P_n = \{x : \tilde{A}x \le \tilde{b}\}$ from Fourier - Motzkin.

what about if

$$P = \operatorname{conv}(x_1, \ldots x_t\}$$

$$P_n = \operatorname{conv}((x_1)_n, \ldots (x_t)_n)$$

get rid of final coordinate.

vertices of $P_n \subseteq \{(x_1)_n, \ldots$

$\ldots (x_t)_n\}$.

$$P = \{x : Ax \le b\}$$

$\downarrow$ Ex.

adding slack:

$$P = \{x : Ax \le b,\ x \geq 0\}$$

$$\downarrow$$

$$Q = \{(x,s) : Ax + Is = b,\ x \geq 0,\ s \geq 0\}.$$

P is projection of Q.

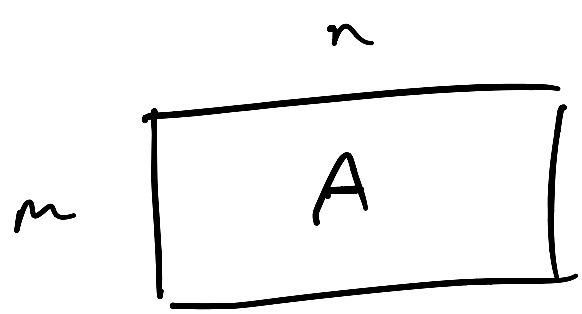
Refresher: BFS

$$P = \{x : Ax = b,\ x \geq 0\}.$$

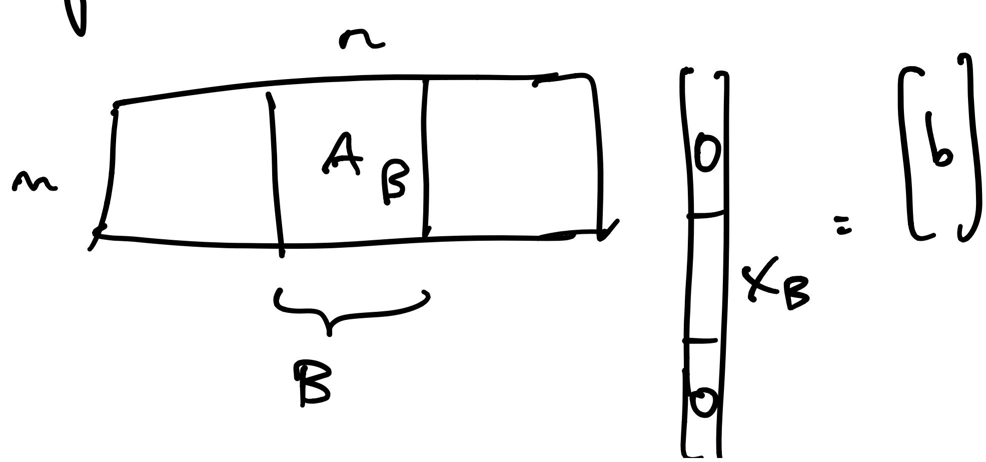
Recall: vertices of P are basic feasible solutions, i.e.

feasible solns obtained as follows:

- Remove redundant rows from A (so that rank $A = m$)



- Choose m columns B of A that form a basis for $\mathbb{R}^m$



- Solve $A_B x_B = b$,

set

$$x^* = \begin{bmatrix} 0 \\ x_B \\ 0 \end{bmatrix} \}B$$

We "proved" in lecture 7 or 8.

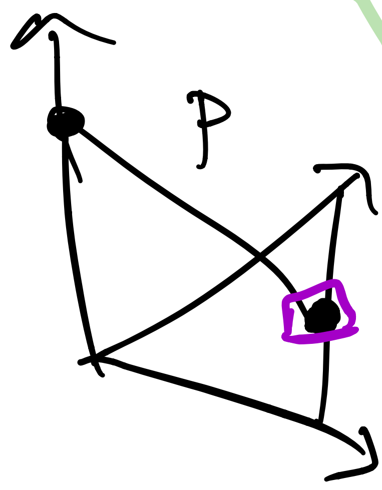
if $x^* \in P$, i.e. x^* feasible,

x^* is bfs.

$\boxed{\text{All vertices of } P \text{ are bfs.}}$

E.g.: in $\mathbb{R}^3$

$$P = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} x \geq 0 \\ y \geq 0,\ z \geq 0 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$



3 potential bfs:

1) $z=0$, $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow (x,y,z) = \boxed{\left(\frac{1}{2}, \frac{1}{2}, 0\right)}$. ✓ feasible,

2) $y=0$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow (x,y,z) = \boxed{(0,0,1)}$ ✓ feasible

3) $x=0$, $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

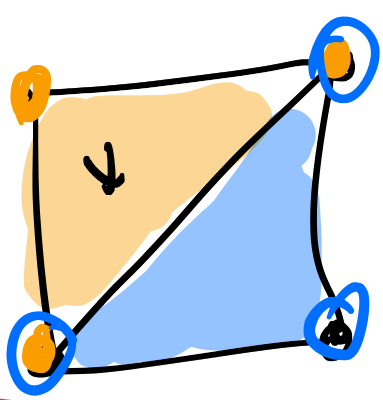
$\Rightarrow (x,y,z) = \boxed{(0,0,1)}$ ✓ feasible.

Another example:

Caratheodory's Theorem

Every point in a polytope $P \subseteq \mathbb{R}^n$ can be written as a convex combination of $\leq n+1$ vertices of P.

(holds for any bounded convex ~~set~~ closed set
vertices ⇝ extreme pts)



Proof Let $x \in P$, $P = \mathrm{conv}(v_1, \ldots, v_t)$.

- consider ways to write x as convex combo of vertices $v_1, \ldots, v_t$ of P.
- Assume affine hull of vertices of P is $\mathbb{R}^n$ (else could translate, rotate P to be $\subseteq \mathbb{R}^{n'}$ for $n' < n$.)

$$Q = \left\{ \lambda : \begin{array}{l} \sum \lambda_i v_i = x \\ \sum \lambda_i = 1 \\ \lambda_i \geq 0 \end{array} \right\}$$

all ways x is convex combo.

$$= \left\{ \lambda : \begin{array}{cc} & t \\ n+1 & \begin{bmatrix} | & & | \\ v_1 & \cdots & v_t \\ | & & | \\ \hline 1 & \cdots & 1 \end{bmatrix} \end{array} \lambda = x, \ \lambda \geq 0 \right\}.$$

$\underbrace{\qquad\qquad}_{A}$

$^{*}\mathrm{aff}(v_1, \dots, v_t) = \mathbb{R}^n \iff \mathrm{rank}\, A = n+1$

(no redundant rows.)

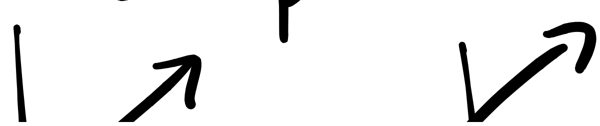
Can use λ^* where λ^* is vertex;

So can take $\lambda^* = \underline{BFS}$!

λ^* has all but $n+1$ coordinates $= 0$! $\square$

<u>Affine hull</u>: X $\mathrm{aff}(X)$ smallest <u>affine</u> space containing it

<u>affine spaces</u> are translations of subspaces.



Total 
unimodularity

- Consider discrete set $X \subseteq \mathbb{R}^n$.

E.g.
- $X \in \mathbb{R}^{n \times n}$ incidence vectors of matchings
- $X \in \mathbb{R}^{n \times n}$ incidence vectors of independent (a.k.a. stable sets) in graph.

- To optimize linear functions over X, enough to do so over $\mathrm{conv}(X)$.

- For this, want simple polyhedral description

$$\boxed{\mathrm{conv}(X) = \{x : Ax \leq b\}. \\ \quad \| \\ \quad P.}$$

Given proposed A, b, how to prove $\mathrm{conv}(X) = P$?

- Easy to show $\mathrm{conv}(X) \subseteq P$; just check $\boxed{Ax \leq b \text{ for all } x \in X.}$

- How about $P \subseteq \mathrm{conv}(X)$?

Harder!

- One way is Algorithmically:

Enough to show $\forall c \in \mathbb{R}^n$

$$\boxed{\max_{x \in X} c^T x = \quad .}$$ ← primal

By weak duality, enough to exhibit dual feasible y and $x \in X$ with

$$\boxed{\quad}$$

e.g. what we did for min-weight perfect matching. (MWPM).

- Today, another way: show extreme points of P integral.

E.g. helps when you know
$$\boxed{X = \{x \in \mathbb{Z}^m : x \in P\}}.$$

i.e.
$X = \{$ feasible set for some integer program. $\}$

$P \subseteq \mathrm{conv}(X)$.

$P = \mathrm{conv}(\text{vertices of } P)$

if vertices of P in $\mathbb{Z}^m$, then they are in X.

Integer program

$$\max c^T x$$
$$x \in P$$
$$x \in \mathbb{Z}^m$$

In this case: L.P. = I.P

$\Updownarrow$

$\boxed{\text{every vertex of } P \text{ is integral}} \iff \boxed{P = \text{conv}(X)}$

$\underbrace{\hphantom{every vertex of P is integral}}$ if this happens, say P

integral.

often not the case $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$x+y=1$

$x+z=1$

$y+z=1$

Method of showing this!

Total Unimodularity

This is true when matrix A

is very <u>special</u>.

Def: matrix A is **totally unimodular** (TU)

if every square submatrix has determinant $\boxed{+1, -1, 0}$.

e.g. $\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, not $\begin{bmatrix} \boxed{2} & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$

Important because:

Theorem: (TU theorem)

Suppose A totally unimodular.

Then $\forall$ integral b,

$$P = \{x : Ax \leq b, x \geq 0\}$$

is integral.

Pre-proof remarks:

- Same proof shows thm also holds for

$$P = \{x : Ax \geq b, x \geq 0\}$$

or

$$P = \{x : Ax = b, x \geq 0\}.$$

- **Converse**: if $P = \{x : Ax \leq b, x \geq 0\}$ is integral for <u>all</u> integral b then A is TU.

(but converse <u>not</u> true for $\{x : Ax = b, x \geq 0\}$).

Proof: First, reduce to equality by adding slack:

let

$$Q = \{(x, s) : Ax + Is = b, x \geq 0, s \geq 0\}.$$

Ex: Q integral $\Leftrightarrow$ P integral.

Fact: $\tilde{A} = [A \mid I]$ TU

$\Leftrightarrow$ A TU.

e.g.

$$\det \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{bmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In general: submatrix of $\tilde{A}$ is cols of A + cols of I; expand down cols I to get determinant of square A submatrix.

Thus: Forget about $\tilde{A}$, assume

$$P = \{x : Ax = b,\ x \geq 0\}.$$

where A is TU.

Recall: vertices are BFS!

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} 0 & A_B & 0 \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ x_B \\ 0 \end{bmatrix} = [b].$$

*

Claim: BFS integral.

we must remove redund. rows

has full rank

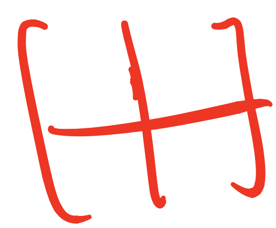
why? $A_B x_B = b$

$$\Rightarrow x_B = A_B^{-1} b$$

But

$$\boxed{x_B = \frac{1}{\det(A_B)} \left[A_B^{adj}\right] b}$$

where A_B^{adj} is



adjugate matrix – entries are subdeterminants.

- in particular A_B^{adj} integral !
- b, A_B^{adj} integral, $|\det A_B| = 1$

$\Rightarrow x_B$ integral
$\Rightarrow$ every vertex is integral. $\square$

Example bipartite matching.

polytope $P \subseteq \mathbb{R}^{n \times n}$ of "fractional matchings" we used for min-weight-perfect-matching:

Recall:

Let (U, V) be bipartition.

$$P := \left\{ X \in \mathbb{R}^{n\times n} : \begin{array}{l} \sum_j x_{ij} = 1 \ \forall i \in U \\ \sum_i x_{ij} = 1 \ \forall j \in V \\ x_{ij} \geq 0 \ \forall i \in U, j \in V. \end{array} \right\}$$

$$:= \{ x : Ax = b, \ x \geq 0 \}.$$

Theorem: The matrix A is totally unimodular.

is totally ...

(what we proved in lecture 2 !).

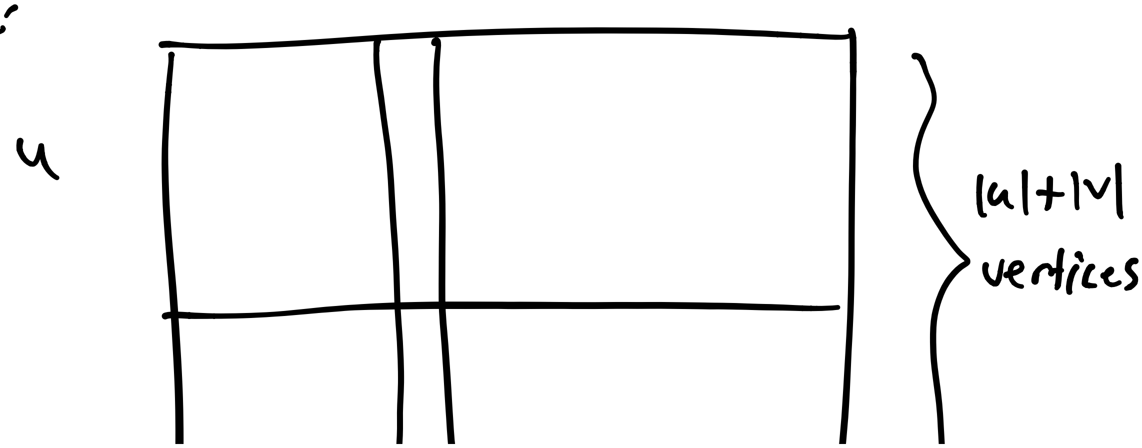
Cor: $\boxed{\text{MWPM} = \min\{c^T x : x \in P\}}$.

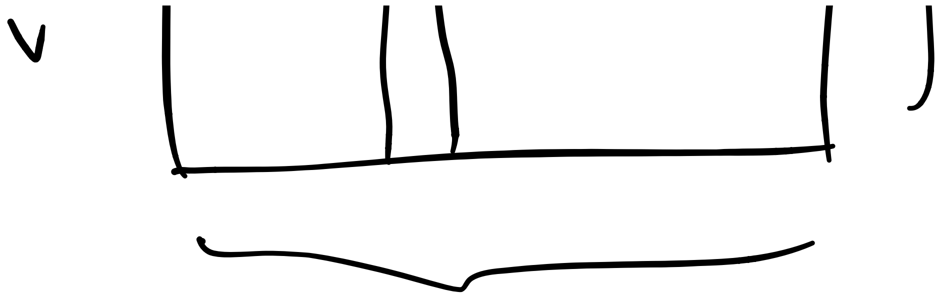
Proof: What's A look like?

A^T is <u>incidence matrix</u> of complete bipartite graph.

<u>i.e.</u>

A =

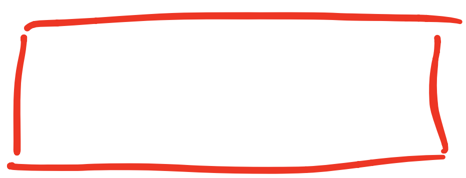




$|U| \cdot |V|$ edges

- To show A is TU, consider square submatrix M & look at cases:

1) if M has 0 row/col,

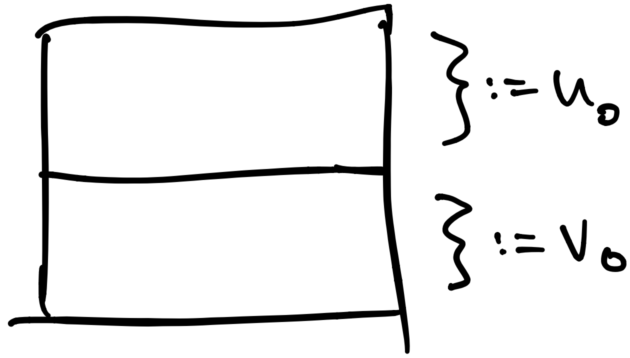


2) if M has row/col w/ only one 1,

3) M has ≥ 2 nonzero entries per row & col.

$\Rightarrow$

M



$$\mathbb{1}_{U_0} = \left[\begin{array}{c} \\ \end{array}\right]\begin{array}{l} U_0 \\ V_0 \end{array} \qquad \mathbb{1}_{V_0} = \left[\begin{array}{c} \\ \end{array}\right]\begin{array}{l} U_0 \\ V_0 \end{array}$$

(add up rows of M in U_0, get $\mathbb{1}^T$).

similarly

$\Rightarrow$ two distinct sol'ns. to ;

rows <u>not</u> lin indep.

$\Rightarrow$

$\square$

P is <u>integral</u> if all its vertices are integral.

$$\Updownarrow$$

$$P = \text{conv}(\text{integer points in } P)$$

$$\Updownarrow$$

L.P $\max c^T x$

$x \in P$

= I.P $\max c^T x$

$x \in P$

$x \in P$

$x \in \mathbb{Z}^n$

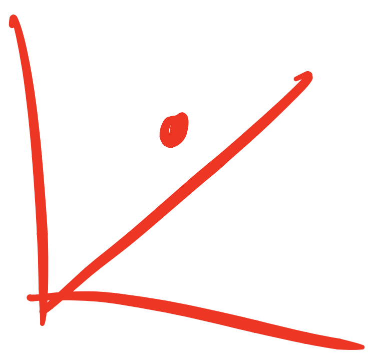
for all $c \in \mathbb{R}^n$

(maximizers same.)

$x + y = 1$

$x + z = 1$

$y + z = 1$



$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

IP infeasible

P not integral, IP infeasible,

LP $c^T(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$